Mathematics: analysis and approaches
Higher level
Paper 3

Name

worked solutions v1

Date: ____________________

1 hour 15 minutes

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

1. [Maximum mark: 30]

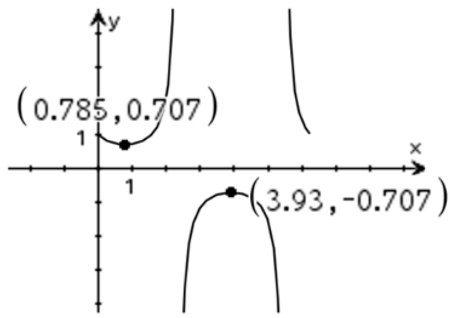
(a) (i) $y = \dfrac{1}{\sin x + \cos x}$ is undefined when $\sin x + \cos x = 0 \;\Rightarrow\; \sin x = -\cos x \;\Rightarrow\; x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

Thus, $c = \dfrac{3\pi}{4}, d = \dfrac{7\pi}{4}$ [2]

(ii) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{\sin x + \cos x}\right) = \dfrac{(\sin x + \cos x)(0) - (\cos x - \sin x)}{(\sin x + \cos x)^2} = \dfrac{\sin x - \cos x}{(\sin x + \cos x)^2}$ ***Q.E.D.*** [3]

(iii) $\dfrac{dy}{dx} = \dfrac{\sin x - \cos x}{(\sin x + \cos x)^2} = 0 \;\Rightarrow\; \sin x - \cos x = 0 \;\Rightarrow\; \sin x = \cos x \;\Rightarrow\; x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$\dfrac{\pi}{4} \approx 0.7854...$ $\quad \dfrac{5\pi}{4} \approx 3.927...$ $\quad y\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sin\left(\dfrac{\pi}{4}\right) + \cos x\left(\dfrac{\pi}{4}\right)} = \dfrac{1}{\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

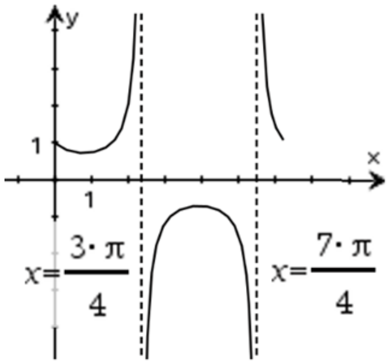


$y\left(\dfrac{5\pi}{4}\right) = \dfrac{1}{\sin\left(\dfrac{5\pi}{4}\right) + \cos x\left(\dfrac{5\pi}{4}\right)} = \dfrac{1}{-\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}} = -\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

Thus, coordinates of S (minimum point) are $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$

and coordinates of T (maximum point) are $\left(\dfrac{5\pi}{4}, -\dfrac{\sqrt{2}}{2}\right)$.

(iv) $y = \dfrac{1}{\sin x + \cos x}$



(b) (i) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{a + b\sin x}{b + a\sin x}\right) = \dfrac{(b + a\sin x)(b\cos x) - (a + b\sin x)(a\cos x)}{(b + a\sin x)^2}$

$= \dfrac{\cos x\left(b^2 + ab\sin x - a^2 - ab\sin x\right)}{(b + a\sin x)^2} = \dfrac{\left(b^2 - a^2\right)\cos x}{(b + a\sin x)^2}$ ***Q.E.D.***

(ii) $0 < a < b \;\Rightarrow\; a \neq b$; $\dfrac{dy}{dx} = \dfrac{\left(b^2 - a^2\right)\cos x}{(b + a\sin x)^2} = 0 \;\Rightarrow\; \cos x = 0 \;\Rightarrow\; x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$y\left(\dfrac{\pi}{2}\right) = \dfrac{a + b}{b + a} = 1$; $y\left(\dfrac{3\pi}{2}\right) = \dfrac{a - b}{b - a} = -1$; thus, point M at $\left(\dfrac{\pi}{2}, 1\right)$ and point N at $\left(\dfrac{3\pi}{2}, -1\right)$

(c) (i) The vertical asymptote of a rational function occurs at a value of x where the function is undefined; i.e. where the denominator, in this case $b+a\sin x$, is zero. Solve $b+a\sin x=0$

$\sin x=-\dfrac{b}{a}$; $0<a<b \Rightarrow -\dfrac{b}{a}<-1$; but $-1\le\sin x\le 1$; $b+a\sin x=0$ has no solutions

Therefore, the graph of f cannot have a vertical asymptote.

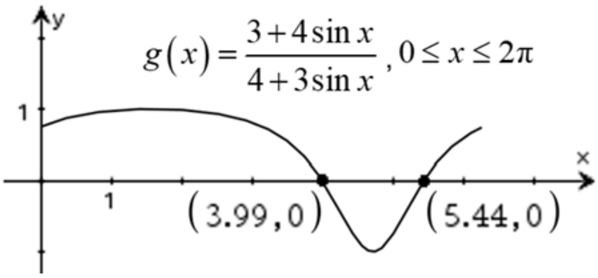
(ii) $\dfrac{dy}{dx}=\dfrac{\left(b^2-a^2\right)\cos x}{\left(b+a\sin x\right)^2}=0 \Rightarrow \cos x=0 \Rightarrow x=\dfrac{\pi}{2},\dfrac{3\pi}{2},\dfrac{5\pi}{2},\dfrac{7\pi}{2},\ldots$

So, horizontal tangents occur where x is equal to a positive odd number times $\dfrac{\pi}{2}$.

A positive odd number can be expressed as $2n-1$ where $\mathbb{Z}^+$.

Thus, the graph of f has a horizontal tangent when $x=(2n-1)\dfrac{\pi}{2}$.

(d) (i)



(ii) total area $=\int_0^{3.99...} g(x)\,dx+\left|\int_{3.99...}^{5.44...} g(x)\,dx\right|\approx 3.2968...+0.84333...$

$\approx 4.1402...\approx 4.14$ square units

2. [Maximum mark: 25]

(a) $\vec{OP}=(1+t)\boldsymbol{i}+(2-2t)\boldsymbol{j}+(3t-1)\boldsymbol{k}$ substituting $t=0$: $\vec{OP}=\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$

Thus, the coordinates of P when $t=0$ are $(1, 2, -1)$.

(b) $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\-1\end{pmatrix}+t\begin{pmatrix}1\\-2\\3\end{pmatrix} \Rightarrow \begin{matrix}x=1+t\\y=2-2t\\z=-1+3t\end{matrix} \Rightarrow \begin{matrix}x-1=t\\ \dfrac{y-2}{-2}=t\\ \dfrac{z+1}{3}=t\end{matrix}$

Thus, Cartesian equations for line L are $x-1=\dfrac{y-2}{-2}=\dfrac{z+1}{3}$.

(c) (i) $\begin{aligned} x &= 1+m \\ y &= 2-2m \\ z &= -1+3m \end{aligned} \Rightarrow 2(1+m)+(2-2m)+(-1+3m)=6 \Rightarrow 3m=3 \Rightarrow m=1$

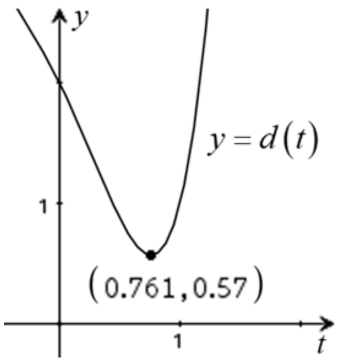
(ii) $\begin{aligned} x &= 1+1 \\ y &= 2-2(1) \\ z &= -1+3(1) \end{aligned} \Rightarrow$ coordinates of P when it lies on plane Π are $(2, 0, 2)$

(iii) The distance travelled is the distance between the points $(1, 2, -1)$ and $(2, 0, 2)$.

Thus, exact distance travelled is $\sqrt{(1-2)^2+(2-0)^2+(-1-2)^2}=\sqrt{14}$ cm^2

(d) (i) The distance, d, from Q to the origin is given by the function

$$d(t)=\sqrt{(t^2-0)^2+(1-t-0)^2+(1-t^2-0)^2}=\sqrt{2t^4-t^2-2t+2}$$



The distance from Q to the origin is a minimum when $t \approx 0.76069... \approx 0.761$ seconds (3 significant figures)

(ii) $\begin{aligned} x &= t^2 \\ y &= 1-t \\ z &= 1-t^2 \end{aligned} \Rightarrow \begin{aligned} x &= (0.76069...)^2 \\ y &= 1-0.76069... \\ z &= 1-(0.76069...)^2 \end{aligned} \Rightarrow \begin{aligned} x &\approx 0.57865... \\ y &\approx 0.23931... \\ z &\approx 0.42135... \end{aligned}$

Coordinates of Q are approximately when min. distance from origin

(e) (i) $\boldsymbol{a}=\begin{pmatrix}0\\1\\1\end{pmatrix}, \boldsymbol{b}=\begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{c}=\begin{pmatrix}4\\-1\\-3\end{pmatrix}$; substituting into the equation $\boldsymbol{a}-\boldsymbol{b}=\lambda(\boldsymbol{b}-\boldsymbol{c})$ gives

$$\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}=\lambda\left[\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}4\\-1\\-3\end{pmatrix}\right] \Rightarrow \begin{pmatrix}-1\\1\\1\end{pmatrix}=\lambda\begin{pmatrix}-3\\1\\3\end{pmatrix} \Rightarrow \lambda=\frac{1}{3} \text{ and } \lambda=1 \text{ which is not possible}$$

Thus, there is no solution for λ. ***Q.E.D.***

(e) continued

(ii) $\overrightarrow{\mathrm{BA}} = \boldsymbol{a} - \boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{\mathrm{CB}} = \boldsymbol{b} - \boldsymbol{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$

Since no λ solves $\boldsymbol{a} - \boldsymbol{b} = \lambda(\boldsymbol{b} - \boldsymbol{c})$ then $\overrightarrow{\mathrm{CB}}$ is not a constant multiple of $\overrightarrow{\mathrm{BA}}$. Hence, $\overrightarrow{\mathrm{BA}}$ and $\overrightarrow{\mathrm{CB}}$ are not parallel which means that the points A, B and C are not collinear and the path of point Q is not a straight line.